Fall 2010

Problem 1 (Sp06, Fa10, Sp13, Sp21). Prove that the intersection of a decreasing sequence of nonempty closed connected subsets of a compact metric space is nonnempty and connected. Give an example to show that this is false if the assumption that the space is compact is dropped.

Problem 2 (Fa88, Fa10). Let R be a finite ring. Prove that there are positive integers m and n with $m > n$ such that $x^m = x^n$ for every x in R.

Problem 3 (Fa10). Suppose that a function f is bounded and analytic on a deleted neighborhood $0 < |z| < \varepsilon$ of the origin. Let

$$f(z) = \sum_{j=-\infty}^{\infty} c_j z^j$$

be the Laurent expansion of f. Show that if j is negative then $c_j = 0$.

Problem 4 (Fa10, Fa17). If the complex conjugate of a complex matrix is equal to its transpose, prove that all its eigenvalues are real.

Problem 5 (Fa10, Sp21). Show that the series $\displaystyle\sum_{n=1}^{\infty} \sin \frac{x}{n^2}$ converges uniformly on any bounded interval in $\mathbb{R}$.

Problem 6 (Fa84, Fa96, Fa10, Fa18). Let gcd abbreviate greatest common divisor and lcm abbreviate least common multiple. For three nonzero integers a, b, c, show that

$$\gcd\{a, \operatorname{lcm}\{b, c\}\} = \operatorname{lcm}\{\gcd\{a, b\}, \gcd\{a, c\}\}.$$

Problem 7 (Fa97, Fa10, Sp24). Evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{\cos kx}{1 + x + x^2}\, \mathrm{d}x$, where $k \geqslant 0$.

Problem 8 (Fa10). Suppose that $f(x_1, \ldots, x_n) = \sum_{ij} a_{ij} x_i x_j$ for some real numbers a_{ij}. If f is nonnegative for all real arguments, show that f can be written as a finite sum of squares of linear forms in $x_1, \ldots, x_n$.

Problem 9 (Sp78, Fa10). 1. For which real numbers $\alpha > 0$ does the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x^{\alpha} \qquad x(0) = 0$$

have a solution on some interval $[0, b], b > 0$?

2. For which values of α are there intervals on which two solutions are defined?

Problem 10 (Fa10). Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous map such that the inverse image of any bounded set is bounded. Show that f achieves either a minimum value or a maximum value.

Problem 11 (Fa10). Show that the splitting field of $x^5 - 10x + 5$ over the rational numbers has Galois group the symmetric group S_5 on 5 letters.

Problem 12 (Fa78, Fa10). Give examples of conformal maps as follows:

1. from $\{z \mid |z| < 1\}$ onto $\{z \mid \Re z < 0\}$,
2. from $\{z \mid |z| < 1\}$ onto itself, with $f(0) = 0$ and $f(1/2) = i/2$,
3. from $\{z \mid |z| < 1\}$ onto the sector $\{z \mid 0 < \arg z < \pi/4\}$,
4. from $\{z \mid z \neq 0, 0 < \arg z < 3\pi/2\}$ onto $\{z \mid z \neq 0, 0 < \arg z < \pi/2\}$.

Problem 13 (Fa10). Find the determinant of the 6×6 matrix with entries $a_{j,k} = j^k$ for $1 \leqslant j, k \leqslant 6$.

Problem 14 (Sp85, Fa10). Let $f(x)$, $0 \leqslant x < \infty$, be continuous, differentiable, with $f(0) = 0$, and that $f'(x)$ is an increasing function of x for $x \geqslant 0$. Prove that

$$g(x) = \begin{cases} f(x)/x & x > 0 \\ f'(0) & x = 0 \end{cases}$$

is an increasing function of x.

Problem 15 (Fa10). Show that there are no simple groups of order 30.

Problem 16 (Fa10, Sp21, Sp26). If f is an analytic function from the unit disc into itself with $f(0) = 0$, prove that $|f'(0)| \leqslant 1$ and find all functions f for which equality holds.

Problem 17 (Fa10). Suppose that f is a positive continuous function on the interval $[a, b]$. Prove that there are polynomials p_n, for $n = 0, 1, 2, \ldots$, such that p_n has highest coefficient x^n, and

$$\int_a^b p_m(x)\, p_n(x)\, f(x)\, \mathrm{d}x = 0$$

if $m \neq n$.

Problem 18 (Fa10). 1. Prove that the series $\frac{0!}{x} - \frac{1!}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \cdots$ diverges for all nonzero x.

2. If $x > 0$ and $G(x) = \int_0^\infty \frac{e^{-tx}}{1+t}\,\mathrm{d}t$ show that the difference between $G(x)$ and the sum of the first n terms of the series has absolute value at most that of the first term omitted.

3. If $x = 100$, prove that the sum of the first 10 terms of the divergent series gives $G(x)$ correctly to more than 10 decimal places.