Fall 2011

Problem 1 (Fa11, Sp21). Find the volume of the solid given by $x^2 + z^2 \leqslant 1$, $y^2 + z^2 \leqslant 1$.

Problem 2 (Fa11). Let $f(x)$ be a irreducible polynomial over the rational numbers $\mathbb{Q}$. Let a in $\mathbb{C}$ be a nonzero complex root of $f(x) = 0$ such that a^2 is also a root. Prove that for some n, $f(x)$ divides $x^n - 1$.

Problem 3 (Sp99, Fa11).

1. Prove that if f is holomorphic on a simply connected region U, and $f(z) \neq 0$ for all $z \in U$, then there is a holomorphic function g on U such that $f(z) = e^{g(z)}$ for all $z \in U$.

2. Does the conclusion of Part 1 remain true if U is replaced by an arbitrary connected open subset of $\mathbb{C}$?

Problem 4 (Fa11). Show that for any integer $n \geqslant 0$ there is a unique polynomial S_n of degree n with real coefficients such that

$$\int_{-1}^{1} S_n(x) P(x)\,\mathrm{d}x = P(1)$$

for all polynomials P of degree at most n. Show that $\int_{-1}^{1} (1-x) S_m(x) S_n(x)\,\mathrm{d}x = 0$ if $m \neq n$.

Problem 5 (Fa11). Suppose given $a_n > 0$ such that $\sum_{n=1}^{\infty} a_n = L < \infty$ and such that for all n, $a_n \leqslant \sum_{m=n+1}^{\infty} a_m$. Show that for all t with $0 < t < L$ there is a subseries a_{n_i} such that $\sum_{i=1}^{\infty} a_{n_i} = t$.

Problem 6 (Fa11). Let G be a group. Show that if G has trivial center then its automorphism group Aut_G has trivial center.

Problem 7 (Fa11). Find the Laurent expansion of

$$f(z) = (1+z)^{-1} + (z^2 - 9)^{-1}$$

in the set $\{z \in \mathbb{C} \mid 1 < |z| < 3\}$.

Problem 8 (Sp83, Fa11, Sp21). Let A be an $n \times n$ real matrix where all entries not on the diagonal are negative, and the sum of the entries in each column is positive. Show that the determinant of A is positive.

Problem 9 (Fa11). The Bessel function $J_1(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ satisfies the differential equation

$$x^2 \frac{\mathrm{d}^2 J_1}{\mathrm{d}x^2} + x \frac{\mathrm{d}J_1}{\mathrm{d}x} + (x^2 - 1) J_1 = 0$$

and also has derivative 1 at 0. Find the coefficients a_n.

Problem 10 (Su80, Sp97, Fa11, Fa13, Fa17, Sp21, Fa25). For each $(a, b, c) \in \mathbb{R}^3$, consider the series

$$\sum_{n=3}^{\infty} \frac{a^n}{n^b \log^c n}$$

Determine the values of (a, b, c) for which the series converges absolutely; converges but not absolutely; diverges.

Problem 11 (Sp81, Fa11). Let $\mathbb{K}$ be one of the fields $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{F}_{4044121}$ (the finite field with $4044121 = 2011^2$ elements). For which of the above choices of $\mathbb{K}$ is the ring $\mathbb{K}[x]/\langle x^4 + 6x - 12 \rangle$ a field?

Problem 12 (Fa11, Sp21). If a and b are points in the open unit disc of the complex plane, show that there is a holomorphic map from the open unit disc to itself with holomorphic inverse that takes a to b.

Problem 13 (Fa81, Fa11). The Fibonacci numbers $F_1, F_2, \ldots$ are defined recursively by $F_1 = 1$, $F_2 = 2$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geqslant 2$. Show that

$$\lim_{n \to \infty} \frac{F_{n+1}}{F_n}$$

exists, and evaluate the limit.
Note: See also **??**.

Problem 14 (Fa11). Prove that a continuous map from a compact metric space to a metric space has closed image.

Problem 15 (Fa11, Sp15, Sp26). Let p be a prime number and G be a group such that $g^p = 1$ for all $g \in G$. Show that if $p = 2$ then G is abelian, and give an example with $p > 2$ where G is not abelian.

Problem 16 (Su77, Fa84, Sp94, Sp96, Sp09, Fa11). Evaluate the integral

$$I(a) = \int_0^{2\pi} \frac{\mathrm{d}\theta}{a + \cos\theta}$$

where a is real and $a > 1$. Why the formula obtained for $I(a)$ is also valid for certain complex (nonreal) values of a?

Problem 17 (Fa87, Fa11). Calculate A^{100} and A^{-7}, where $A = \begin{pmatrix} 3/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$.

Problem 18 (Sp02, Fa11). Let (X, ρ) and (Y, σ) be metric spaces. Assume that:

- $f, f_1, f_2, \ldots$ are bijective functions of X onto Y with inverses $g, g_1, g_2, \ldots$
- g is uniformly continuous
- $f_n \to f$ uniformly as $n \to \infty$.

Prove that $g_n \to g$ uniformly as $n \to \infty$.