

Fall 2016

Problem 1 (Fa16). 1. Prove that if $s > 1$ then $\sum_{n \geq 1} n^{-s} = \prod_p (1 - p^{-s})^{-1}$, where the product is over all primes p .

2. Prove that the sum $\sum_p 1/p$ over all primes p diverges.

Problem 2 (Fa16). Let $x : [a, b] \rightarrow \mathbb{R}$ and $f : [a, b] \rightarrow \mathbb{R}$ be nonnegative continuous functions satisfying

$$x^2(t) \leq 1 + \int_a^t f(s)x(s) \, ds$$

for $a \leq t \leq b$. Show that

$$x(t) \leq 1 + \frac{1}{2} \int_a^t f(s) \, ds$$

for $a \leq t \leq b$.

Problem 3 (Fa16). Given $K \geq 0$, let Lip_K be the set of functions $f : \mathbb{R} \rightarrow \mathbb{R}$ which satisfy $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in \mathbb{R}$.

- Show that the formula $d(f_1, f_2) = \sum_{j=1}^{\infty} 2^{-j} \sup_{z \in [-j, j]} |f_1(z) - f_2(z)|$ converges and defines a metric d on Lip_K .
- Show that Lip_K is a complete metric space with this metric.

Problem 4 (Fa82, Sp92, Fa16). Evaluate $\int_{-\infty}^{\infty} \frac{\sin^3 x}{x^3} \, dx$.

Problem 5 (Fa96, Fa16). Does there exist a function f , analytic in the punctured plane $\mathbb{C} \setminus \{0\}$, such that

$$|f(z)| \geq \frac{1}{\sqrt{|z|}}$$

for all nonzero z ?

Problem 6 (Sp15, Fa16, Sp17). Let $\mathbb{D}$ be the unit disc in the complex plane $\mathbb{C}$, $f : \mathbb{D} \rightarrow \mathbb{C}$ an analytic function with

$$|f^{(k)}(0)| \leq M$$

for all $k \geq 0$, and let $t_i, s_i \in \mathbb{D}$ for $1 \leq i \leq n$. Define the matrix $A_{ij} = f(t_i s_j)$ for $1 \leq i, j \leq n$. For each $r \geq 1$ find an $n \times n$ matrix B with rank $\leq r$ and

$$|A_{ij} - B_{ij}| \leq \frac{2M}{r!}$$

for $1 \leq i, j \leq n$.

Problem 7 (Su81, Fa16). If two rational matrices are similar over $\mathbb{C}$, then they are similar over $\mathbb{Q}$.

Problem 8 (Fa16). Let $M_2(\mathbb{Q})$ be the ring of all 2×2 matrices with coefficients in $\mathbb{Q}$. Describe all field extensions $\mathbb{K}$ of $\mathbb{Q}$ such that there is an injective ring homomorphism $\mathbb{K} \rightarrow M_2(\mathbb{Q})$.

Problem 9 (Su79, Sp82, Sp83, Su84, Fa91, Fa98, Sp11, Fa15, Fa16, Fa23). Let $\mathbb{F}$ be a finite field with q elements and let V be an n -dimensional vector space over $\mathbb{F}$.

1. Determine the number of elements in V .
2. Let $\text{GL}_n(\mathbb{F})$ denote the group of all $n \times n$ nonsingular matrices over $\mathbb{F}$. Determine the order of $\text{GL}_n(\mathbb{F})$.
3. Let $\text{SL}_n(\mathbb{F})$ denote the subgroup of $\text{GL}_n(\mathbb{F})$ consisting of matrices with determinant 1. Find the order of $\text{SL}_n(\mathbb{F})$.

Problem 10 (Fa16). Let $C = \int_{-\infty}^{\infty} e^{-x^2} dx$ and let S_n be the $(n-1)$ -dimensional “surface area” of the unit sphere in $\mathbb{R}^n$, so $S_2 = 2\pi$, $S_3 = 4\pi/3, \dots$

1. Prove that $C^n = S_n \Gamma(n/2)/2$, where $\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} dt$ is the Gamma function.
2. Evaluate C .
3. Evaluate S_4 .

Problem 11 (Fa16). Let K be a compact subset of $\mathbb{R}^n$ and $f(x) = d(x, K)$ be the Euclidean distance from x to the nearest point of K .

- Show that f is continuous and $f(x) = 0$ if $x \in K$.
- Let $g(x) = \max\{1 - f(x), 0\}$. Show that $\int g^m$ converges to the n -dimensional volume of K as $m \rightarrow \infty$.

Problem 12 (Fa16). 1. Suppose that I is a closed interval and f is a smooth function from I to I such that $|f'|$ is bounded by some number $r < 1$ on I . Let a_0 be in I and put $a_{n+1} = f(a_n)$. Prove that the sequence (a_n) tends to the unique root of $f(x) = x$ in I .

2. Show that if a_0 is real and $a_{n+1} = \cos a_n$ then a_n tends to a root of $\cos x = x$.

Problem 13 (Fa16). Let $f(z) = z(e^z - 1)$. Prove there exists an analytic function $h(z)$ defined near $z = 0$ such that $f(z) = h(z)^2$. Find the first 3 terms in the power series expansion $h(z) = \sum a_n z^n$. Does $h(z)$ extend to an entire function on $\mathbb{C}$?

Problem 14 (Fa16). Let $f_t(z)$ be a family of entire functions depending analytically on t in the open unit disc $\mathbb{D}$ in $\mathbb{C}$. Suppose that for all t , $f_t(z)$ is nonvanishing on the unit circle S^1 in $\mathbb{C}$. Prove that for each $k \geq 0$,

$$N_k(t) = \sum_{\{|z| < 1 \mid f_t(z) = 0\}} z^k$$

where the zeros of $f_t(z)$ are taken with multiplicity in the sum, is an analytic function of t .

Problem 15 (Fa16). Let A be an $m \times n$ matrix of rank r and B a $p \times q$ matrix of rank s . Find the dimension of the vector space of $n \times p$ matrices X such that $AXB = 0$.

Problem 16 (Fa16). Find an example of a vector space V over the real numbers $\mathbb{R}$ and two linear maps $f, g : V \rightarrow V$ such that f is injective but not surjective and g is surjective but not injective and such that $f + g$ is equal to the identity map 1_V .

Problem 17 (Fa16). Let G be a group and n be a positive integer. Assume that there exists a surjective group homomorphism $\mathbb{Z}^n \rightarrow G$ and an injective group homomorphism $\mathbb{Z}^n \rightarrow G$. Prove that the group G is isomorphic to $\mathbb{Z}^n$.

Problem 18 (Fa16). Find the number of groups of order 12 up to isomorphism.