

Fall 1979

Problem 1 (Fa79, Fa89). Prove that the polynomial

$$p(z) = z^{47} - z^{23} + 2z^{11} - z^5 + 4z^2 + 1$$

has at least one zero in the disc $|z| < 1$.

Problem 2 (Fa79, Fa90). Suppose that f is analytic on the open upper half-plane and satisfies $|f(z)| \leq 1$ for all z , $f(i) = 0$. How large can $|f(2i)|$ be under these conditions?

Problem 3 (Fa79, Sp88, Fa91). Prove that every finite group of order > 2 has a nontrivial automorphism.

Problem 4 (Fa79, Su80, Fa82). Consider the polynomial ring $\mathbb{Z}[x]$ and the ideal $\mathfrak{J}$ generated by 7 and $x - 3$.

1. Show that for each $r \in \mathbb{Z}[x]$, there is an integer α satisfying $0 \leq \alpha \leq 6$ such that $r - \alpha \in \mathfrak{J}$.
2. Find α in the special case $r = x^{250} + 15x^{14} + x^2 + 5$.

Problem 5 (Fa79). Let A be a real *skew-symmetric* matrix ($A_{ij} = -A_{ji}$). Prove that A has even rank.

Problem 6 (Fa79). Let N be a linear operator on an n -dimensional vector space, $n > 1$, such that $N^n = 0$, $N^{n-1} \neq 0$. Prove there is no operator X with $X^2 = N$.

Problem 7 (Fa79, Su81, Fa19). Let V be the vector space of sequences (a_n) of complex numbers. The shift operator $S : V \rightarrow V$ is defined by

$$S((a_1, a_2, a_3, \dots)) = (a_2, a_3, a_4, \dots)$$

1. Find the eigenvectors of S .
2. Show that the subspace W consisting of the sequences (x_n) with $x_{n+2} = x_{n+1} + x_n$ is a two-dimensional, S -invariant subspace of V and exhibit an explicit basis for W .
3. Find an explicit formula for the n -th Fibonacci number F_n , where $F_2 = F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$.

Note: See also ??.

Problem 8 (Fa79). Prove that

$$\lim_{n \rightarrow \infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right) = \log 2$$

Problem 9 (Fa79). Given that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$ find $f'(t)$ explicitly, where

$$f(t) = \int_{-\infty}^{\infty} e^{-tx^2} dx \quad t > 0$$

Problem 10 (Su79, Fa79, Fa82, Su85). Find all pairs of C^∞ functions $x(t)$ and $y(t)$ on $\mathbb{R}$ satisfying

$$x'(t) = 2x(t) - y(t) \quad y'(t) = x(t)$$

Problem 11 (Fa79). An accurate map of California is spread out flat on a table in Evans Hall, in Berkeley. Prove that there is exactly one point on the map lying directly over the point it represents.

Problem 12 (Fa79). Consider the following properties of a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$:

1. f is continuous.
2. The graph of f is connected in $\mathbb{R}^n \times \mathbb{R}$.

Prove or disprove the implications $1 \Rightarrow 2, 2 \Rightarrow 1$.

Problem 13 (Fa79, Fa80). Let $\{P_n\}$ be a sequence of real polynomials of degree $\leq D$, a fixed integer. Suppose that $P_n(x) \rightarrow 0$ pointwise for $0 \leq x \leq 1$. Prove that $P_n \rightarrow 0$ uniformly on $[0, 1]$.

Problem 14 (Fa79, Su81, Fa92). Let $y = y(x)$ be a solution of the differential equation $y'' = -|y|$ with $-\infty < x < \infty$, $y(0) = 1$ and $y'(0) = 0$.

1. Show that y is an even function.
2. Show that y has exactly one zero on the positive real axis.

Problem 15 (Fa79, Su81, Sp14). Suppose f and g are entire functions with $|f(z)| \leq |g(z)|$ for all z . Prove that $f(z) = cg(z)$ for some constant c .

Problem 16 (Fa79, Fa80, Sp85, Su85, Fa96, Fa98, Sp99, Sp26). Prove that

$$\int_0^{\infty} \frac{x^{\alpha-1}}{1+x} dx = \frac{\pi}{\sin \pi \alpha}$$

What restrictions must be placed on α ?

Problem 17 (Fa79). Let A be an $n \times n$ complex matrix. Prove there is a unitary matrix U such that $T = UAU^{-1}$ is upper triangular, that is, $T_{jk} = 0$ for $j > k$.

Problem 18 (Fa79). Let B denote the matrix

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

where a , b , and c are real and $|a|$, $|b|$, and $|c|$ are distinct. Show that there are exactly four symmetric matrices of the form BQ , where Q is a real orthogonal matrix of determinant 1.

Problem 19 (Fa79, Fa87, Sp10, Fa15). Let $M_{n \times n}(\mathbb{F})$ be the ring of $n \times n$ matrices over a field $\mathbb{F}$. Prove that it has no 2-sided ideals except $M_{n \times n}(\mathbb{F})$ and $\{0\}$.

Problem 20 (Fa79, Fa82). Let G be the abelian group defined by generators x , y , and z , and relations

$$15x + 3y = 0$$

$$3x + 7y + 4z = 0$$

$$18x + 14y + 8z = 0$$

1. Express G as a direct product of two cyclic groups.
2. Express G as a direct product of cyclic groups of prime power order.
3. How many elements of G have order 2?