

Fall 1981

Problem 1 (Sp77, Fa81, Sp82, Sp26). Evaluate $\int_{-\infty}^{\infty} \frac{\cos nx}{x^4 + 1} dx$.

Problem 2 (Fa81). Consider an autonomous system of differential equations

$$\frac{dx_i}{dt} = F_i(x_1, \dots, x_n)$$

where $F = (F_1, \dots, F_n) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a C^1 vector field.

1. Let u and v be two solutions on $a < t < b$. Assuming that

$$\langle D F(x)z, z \rangle \leq 0$$

for all x, z in $\mathbb{R}^n$, show that $\|u(t) - v(t)\|^2$ is a decreasing function of t .

2. Let $w(t)$ be a solution defined for $t > 0$. Assuming that

$$\langle D F(x)z, z \rangle \leq -\|z\|^2$$

show that there exists $c \in \mathbb{R}^n$ such that $\lim_{t \rightarrow \infty} w(t) = c$

Problem 3 (Fa81, Sp95). Let G be a subgroup of S_n of order p^k , where p is a prime not dividing n . Show that G has a *fixed point*; that is, one of the objects is left fixed by every element of G .

Problem 4 (Fa81, Sp13, Fa22). Prove the following three statements about real $n \times n$ matrices.

1. If A is an orthogonal matrix whose eigenvalues are all different from -1 , then $I + A$ is nonsingular and $S = (I - A)(I + A)^{-1}$ is skew-symmetric.
2. If S is a skew-symmetric matrix, then $A = (I - S)(I + S)^{-1}$ is an orthogonal matrix with no eigenvalue equal to -1 .
3. The correspondence $A \leftrightarrow S$, called the Cayley transform, from Parts 1 and 2 is one-to-one.

Problem 5 (Fa81, Fa11). The Fibonacci numbers $F_1, F_2, \dots$ are defined recursively by $F_1 = 1$, $F_2 = 2$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$. Show that

$$\lim_{n \rightarrow \infty} \frac{F_{n+1}}{F_n}$$

exists, and evaluate the limit.

Note: See also ??.

Problem 6 (Fa81). Let f and g be continuous functions on $\mathbb{R}$ such that $f(x+1) = f(x)$, $g(x+1) = g(x)$, for all $x \in \mathbb{R}$. Prove that

$$\lim_{n \rightarrow \infty} \int_0^1 f(x)g(nx) \, dx = \int_0^1 f(x) \, dx \int_0^1 g(x) \, dx$$

Problem 7 (Su77, Fa81, Sp92, Fa08). Prove that $\alpha = \sqrt{5} + \sqrt{7}$ is algebraic over $\mathbb{Q}$ by writing down a polynomial $f(x) \in \mathbb{Q}[x]$ of degree 4 having α as a zero. Show that $f(x)$ is irreducible over $\mathbb{Q}$.

Problem 8 (Fa81, Fa17, Fa22). 1. How many zeros does the function $f(z) = 3z^{100} - e^z$ have inside the unit circle (counting multiplicities)?

2. Are the zeros distinct?

Problem 9 (Fa81, Sp24). Let $A, B \in M_{2 \times 2}$ be given by:

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 1 \\ 0 & 4 \end{pmatrix}$$

Define a linear transformation $L : M_{2 \times 2} \rightarrow M_{2 \times 2}$ by $L(X) = AXB$. Compute the trace and the determinant of L .

Problem 10 (Fa81). Let $A = (a_{ij})$ be an $n \times n$ matrix whose entries a_{ij} are real valued differentiable functions defined on $\mathbb{R}$. Assume that the determinant $\det A$ of A is everywhere positive. Let $B = (b_{ij})$ be the inverse matrix of A . Prove the formula

$$\frac{d}{dt} \log(\det A) = \sum_{i,j=1}^n \frac{da_{ij}}{dt} b_{ji}$$

Problem 11 (Fa81). Consider the complex 3×3 matrix

$$A = \begin{pmatrix} a_0 & a_1 & a_2 \\ a_2 & a_0 & a_1 \\ a_1 & a_2 & a_0 \end{pmatrix}$$

where $a_0, a_1, a_2 \in \mathbb{C}$.

1. Show that $A = a_0 I_3 + a_1 E + a_2 E^2$, where $E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$
2. Use Part 1 to find the complex eigenvalues of A .
3. Generalize Parts 1 and 2 to $n \times n$ matrices.

Problem 12 (Fa77, Fa81, Fa21). Let $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the function defined by $u(x, y) = x^3 - 3xy^2$. Show that u is harmonic and find $v : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that the function $f : \mathbb{C} \rightarrow \mathbb{C}$ defined by

$$f(x + iy) = u(x, y) + iv(x, y)$$

is analytic.

Problem 13 (Fa81). Let f be a real valued function on $\mathbb{R}^n$ of class $\mathcal{C}^2$. A point $x \in \mathbb{R}^n$ is a *critical point* of f if all the partial derivatives of f vanish at x ; a critical point is *nondegenerate* if the $n \times n$ matrix

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \right)$$

is nonsingular.

Let x be a nondegenerate critical point of f . Prove that there is an open neighborhood of x which contains no other critical points, i.e., the nondegenerate critical points are isolated.

Problem 14 (Fa81). Let $V : \mathbb{R}^n \rightarrow \mathbb{R}$ be a $\mathcal{C}^1$ function and consider the system of second order differential equations

$$x_i''(t) = f_i(x(t)) \quad 1 \leq i \leq n$$

where

$$f_i = -\frac{\partial V}{\partial x_i}$$

Let $x(t) = (x_1(t), \dots, x_n(t))$ be a solution of this system on a finite interval $a < t < b$.

1. Show that the function

$$H(t) = \frac{1}{2} \langle x'(t), x'(t) \rangle + V(x(t))$$

is constant for $a < t < b$.

2. Assuming that $V(x) \geq M > -\infty$ for all $x \in \mathbb{R}^n$, show that $x(t)$, $x'(t)$, and $x''(t)$ are bounded on $a < t < b$, and then prove all three limits

$$\lim_{t \rightarrow b} x(t) \quad \lim_{t \rightarrow b} x'(t) \quad \lim_{t \rightarrow b} x''(t)$$

exist.

Problem 15 (Fa81, Sp89, Fa97). Let f be a holomorphic map of the unit disc $\mathbb{D} = \{z \mid |z| < 1\}$ into itself, which is not the identity map $f(z) = z$. Show that f can have, at most, one fixed point.

Problem 16 (Sp79, Fa81). Let G be a group with three normal subgroups N_1 , N_2 , and N_3 . Suppose $N_i \cap N_j = \{e\}$ and $N_i N_j = G$ for all i, j with $i \neq j$. Show that G is abelian and N_i is isomorphic to N_j for all i, j .

Problem 17 (Fa81, Fa85). Let f be a continuous function on $[0, 1]$. Evaluate the following limits.

$$\lim_{n \rightarrow \infty} \int_0^1 x^n f(x) \, dx$$

$$\lim_{n \rightarrow \infty} (n+1) \int_0^1 x^n f(x) \, dx$$

Problem 18 (Fa81, Sp84, Sp23). If two real matrices are similar over $\mathbb{C}$, then they are similar over $\mathbb{R}$.

Problem 19 (Fa81). Either prove or disprove (by a counterexample) each of the following statements:

1. Let $f : \mathbb{R} \rightarrow \mathbb{R}$, $g : \mathbb{R} \rightarrow \mathbb{R}$ be such that $\lim_{t \rightarrow a} g(t) = b$ and $\lim_{t \rightarrow b} f(t) = c$. Then $\lim_{t \rightarrow a} f(g(t)) = c$.
2. If $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and U is an open set in $\mathbb{R}$, then $f(U)$ is an open set in $\mathbb{R}$.
3. Let f be of class $\mathcal{C}^\infty$ on the interval $(-1, 1)$. Suppose that $|f^{(n)}(x)| \leq 1$ for all $n \geq 1$ and all x in the interval. Then f is *real analytic*, that is, it has a convergent power series expansion in a neighborhood of each point of the interval.

Problem 20 (Sp78, Fa81). Let G be a group of order 10 which has a normal subgroup of order 2. Prove that G is abelian.