

Fall 1984

Problem 1 (Fa84). Let G be a group and H a subgroup of index $n < \infty$. Prove or disprove the following statements:

1. If $a \in G$, then $a^n \in H$.
2. If $a \in G$, then for some k , $1 \leq k \leq n$, we have $a^k \in H$.

Problem 2 (Fa84). Let A and B be $n \times n$ real matrices, and k a positive integer. Find

1.

$$\lim_{t \rightarrow 0} \frac{1}{t} \left((A + tB)^k - A^k \right)$$

2.

$$\left. \frac{d}{dt} \operatorname{tr}(A + tB)^k \right|_{t=0}$$

Problem 3 (Fa84). Prove or supply a counterexample: If f is a nondecreasing real valued function on $[0, 1]$, then there is a sequence of continuous functions on $[0, 1]$, $\{f_n\}$, such that for each $x \in [0, 1]$,

$$\lim_{n \rightarrow \infty} f_n(x) = f(x)$$

Problem 4 (Fa84, Fa18). Evaluate $\int_0^\infty \frac{x - \sin x}{x^3} dx$.

Problem 5 (Fa84). Consider the differential equation $\frac{dy}{dx} = 3xy + \frac{y}{1 + y^2}$. Prove

1. For each $n = 1, 2, \dots$, there is a unique solution $y = f_n(x)$ defined for $0 \leq x \leq 1$ such that $f_n(0) = 1/n$.
2. $\lim_{n \rightarrow \infty} f_n(1) = 0$.

Problem 6 (Fa84, Fa96, Fa10, Fa18). Let gcd abbreviate greatest common divisor and lcm abbreviate least common multiple. For three nonzero integers a, b, c , show that

$$\gcd\{a, \operatorname{lcm}\{b, c\}\} = \operatorname{lcm}\{\gcd\{a, b\}, \gcd\{a, c\}\}.$$

Problem 7 (Fa84, Fa23). Let $\mathbb{R}[x_1, \dots, x_n]$ be the polynomial ring over the real field $\mathbb{R}$ in the n variables $x_1, \dots, x_n$. Let the matrix A be the $n \times n$ matrix whose i -th row is $(1, x_i, x_i^2, \dots, x_i^{n-1})$, $i = 1, \dots, n$. Show that

$$\det A = \prod_{i>j} (x_i - x_j)$$

Problem 8 (Fa84). Let a, b, c , and d be real numbers, not all zero. Find the eigenvalues of the following 4×4 matrix and describe the eigenspace decomposition of $\mathbb{R}^4$:

$$\begin{pmatrix} aa & ab & ac & ad \\ ba & bb & bc & bd \\ ca & cb & cc & cd \\ da & db & dc & dd \end{pmatrix}$$

Problem 9 (Fa84). Let f and g be analytic functions in the open unit disc, and let C_r denote the circle with center 0 and radius r , oriented counterclockwise.

1. Prove that the integral

$$\frac{1}{2\pi i} \int_{C_r} \frac{1}{w} f(w) g\left(\frac{z}{w}\right) dw$$

is independent of r as long as $|z| < r < 1$ and that it defines an analytic function $h(z)$, $|z| < 1$.

2. Prove or supply a counterexample: If $f \not\equiv 0$ and $g \not\equiv 0$, then $h \not\equiv 0$.

Problem 10 (Fa83, Fa84). Prove or supply a counterexample: If f and g are C^1 real valued functions on $(0, 1)$, if

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} g(x) = 0$$

if g and g' never vanish, and if

$$\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = c$$

then

$$\lim_{x \rightarrow 0} \frac{f'(x)}{g'(x)} = c$$

Problem 11 (Fa84). Show that all groups of order ≤ 5 are abelian. Give an example of a non-abelian group of order 6.

Problem 12 (Fa84). Let θ and φ be fixed, $0 \leq \theta \leq 2\pi$, $0 \leq \varphi \leq 2\pi$ and let R be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ whose matrix in the standard basis $\vec{i}$, $\vec{j}$, and $\vec{k}$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix}$$

Let S be the linear transformation of $\mathbb{R}^3$ to $\mathbb{R}^3$ whose matrix with respect to the basis

$$\left\{ \frac{1}{\sqrt{2}}(\vec{i} + \vec{k}), \vec{j}, \frac{1}{\sqrt{2}}(\vec{i} - \vec{k}) \right\}$$

is

$$\begin{pmatrix} \cos \varphi & \sin \varphi & 0 \\ -\sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Prove that $T = R \circ S$ leaves a line invariant.

Problem 13 (Fa84). Show that if f is a homeomorphism of $[0, 1]$ onto itself, then there is a sequence $\{p_n\}$, $n = 1, 2, 3, \dots$, of polynomials such that $p_n \rightarrow f$ uniformly on $[0, 1]$ and each p_n is a homeomorphism of $[0, 1]$ onto itself.

Problem 14 (Fa84). Consider the differential equation

$$\frac{dx}{dt} = y \quad \frac{dy}{dt} = -ay - x^3 - x^5 \quad a > 0$$

1. Show that

$$F(x, y) = \frac{y^2}{2} + \frac{x^4}{4} + \frac{x^6}{6}$$

decreases along solutions.

2. Show that for any $\varepsilon > 0$, there is a $\delta > 0$ such that whenever $\|(x(0), y(0))\| \leq \delta$, there is a unique solution $(x(t), y(t))$ of the given equations with the initial condition $(x(0), y(0))$ which is defined for all $t \geq 0$ and satisfies $\|(x(t), y(t))\| < \varepsilon$.

Problem 15 (Fa84). Let a be an element in a field $\mathbb{F}$ and let p be a prime. Assume a is not a p -th power. Show that the polynomial $x^p - a$ is irreducible in $\mathbb{F}[x]$.

Problem 16 (Fa78, Fa84). Let M be the $n \times n$ matrix over a field $\mathbb{F}$, all of whose entries are equal to 1.

1. Find the characteristic polynomial of M .
2. Is M diagonalizable?
3. Find the Jordan Canonical Form of M . Does the Jordan form depend on the characteristic of the field $\mathbb{F}$.

Problem 17 (Fa84). Let P_n be the vector space of all real polynomials with degrees at most n . Let $D : P_n \rightarrow P_n$ be given by differentiation: $D(p) = p'$. Let π be a real polynomial. What is the minimal polynomial of the transformation $\pi(D)$?

Problem 18 (Fa84). Prove or supply a counterexample: If f is a continuous complex valued function defined on a connected open subset of the complex plane and if f^2 is analytic, then f is analytic.

Problem 19 (Fa84, Fa97). Let f be a $\mathcal{C}^2$ function on the real line. Assume f is bounded with bounded second derivative. Let

$$A = \sup_{x \in \mathbb{R}} |f(x)| \quad \text{and} \quad B = \sup_{x \in \mathbb{R}} |f''(x)|$$

Prove that

$$\sup_{x \in \mathbb{R}} |f'(x)| \leq 2\sqrt{AB}$$

Problem 20 (Fa84). Evaluate $\int_{-\infty}^{\infty} \frac{dx}{(1+x+x^2)^2}$.

Problem 21 (Su77, Fa84, Sp94, Sp96, Sp09, Fa11). Evaluate the integral

$$I(a) = \int_0^{2\pi} \frac{d\theta}{a + \cos \theta}$$

where a is real and $a > 1$. Why the formula obtained for $I(a)$ is also valid for certain complex (nonreal) values of a ?