

Fall 1987

Problem 1 (Fa87). Prove that $\cos^p \theta \leq \cos p\theta$ for $0 \leq \theta \leq \pi/2$ and $0 < p < 1$.

Problem 2 (Fa87). Suppose that $\{f_n\}$ is a sequence of nondecreasing functions which map the unit interval into itself. Suppose that

$$\lim_{n \rightarrow \infty} f_n(x) = f(x)$$

pointwise and that f is a continuous function. Prove that $f_n(x) \rightarrow f(x)$ uniformly as $n \rightarrow \infty$, $0 \leq x \leq 1$.

Problem 3 (Fa87). Show that the following limit exists and is finite:

$$\lim_{t \rightarrow 0^+} \left(\int_0^1 \frac{dx}{(x^4 + t^4)^{1/4}} + \log t \right)$$

Problem 4 (Fa87). Let u and v be two real valued C^1 functions on $\mathbb{R}^2$ such that the gradient ∇u is never 0, and such that, at each point, ∇v and ∇u are linearly dependent vectors. Given $p_0 = (x_0, y_0) \in \mathbb{R}^2$, show that there is a C^1 function F of one variable such that $v(x, y) = F(u(x, y))$ in some neighborhood of p_0 .

Problem 5 (Fa87, Fa11). Calculate A^{100} and A^{-7} , where $A = \begin{pmatrix} 3/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$.

Problem 6 (Fa87, Sp96). Let G and H be finite groups of relatively prime order. Show that $\text{Aut}_{G \times H} \simeq \text{Aut}_G \times \text{Aut}_H$.

Problem 7 (Fa87). Let A and B be real $n \times n$ symmetric matrices with B positive definite. Consider the function defined for $x \neq 0$ by

$$G(x) = \frac{\langle Ax, x \rangle}{\langle Bx, x \rangle}$$

- Show that G attains its maximum value.
- Show that any maximum point U for G is an eigenvector for a certain matrix related to A and B and show which matrix.

Problem 8 (Su78, Fa87, Sp93). Let R be the set of 2×2 matrices of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

where a, b are elements of a given field $\mathbb{F}$. Show that with the usual matrix operations, R is a commutative ring with identity. For which of the following fields $\mathbb{F}$ is R a field: $\mathbb{F} = \mathbb{Q}, \mathbb{C}, \mathbb{Z}_5, \mathbb{Z}_7$?

Problem 9 (Fa87, Sp16). Evaluate the integral $I = \int_0^{2\pi} \frac{\cos^2 3\theta}{5 - 4 \cos 2\theta} d\theta$.

Problem 10 (Fa87, Sp19). If $f(z)$ is analytic in the open disc $|z| < 1$, and $|f(z)| \leq \frac{1}{1 - |z|}$. Show that

$$\left| \frac{f^{(n)}(0)}{n!} \right| \leq (n+1) \left(1 + \frac{1}{n} \right)^n < e(n+1)$$

Problem 11 (Fa87). Let V be a finite dimensional vector space and $T : V \rightarrow V$ a diagonalizable linear transformation. Let $W \subset V$ be a linear subspace which is mapped into itself by T . Show that the restriction of T to W is diagonalizable.

Problem 12 (Fa87). Let $F \subset E$ be an infinite field and an extension. If two matrices in F are similar over E then they are similar over F .

Problem 13 (Fa87, Fa93). Let A be the group of rational numbers under addition, and let M be the group of positive rational numbers under multiplication. Determine all homomorphisms $\varphi : A \rightarrow M$.

Problem 14 (Fa79, Fa87, Sp10, Fa15). Let $M_{n \times n}(\mathbb{F})$ be the ring of $n \times n$ matrices over a field $\mathbb{F}$. Prove that it has no 2-sided ideals except $M_{n \times n}(\mathbb{F})$ and $\{0\}$.

Problem 15 (Fa87). Let $f(z)$ be analytic for $z \neq 0$, and suppose that $f(1/z) = f(z)$. Suppose also that $f(z)$ is real for all z on the unit circle $|z| = 1$. Prove that $f(z)$ is real for all real $z \neq 0$.

Problem 16 (Su80, Fa83, Sp84, Fa87, Fa96, Fa07). How many zeros does the complex polynomial

$$3z^9 + 8z^6 + z^5 + 2z^3 + 1$$

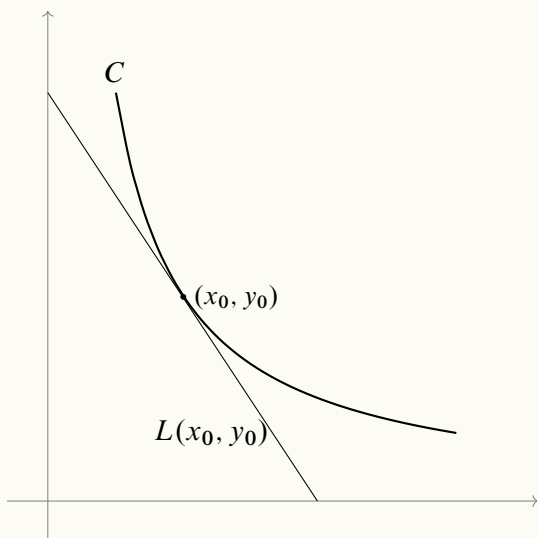
have in the annulus $1 < |z| < 2$?

Problem 17 (Fa87). Let u be a positive *harmonic* function on $\mathbb{R}^2$; that is,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Show that u is constant.

Problem 18 (Fa87). Find a curve C in $\mathbb{R}^2$, passing through the point $(3, 2)$, with the following property: Let $L(x_0, y_0)$ be the segment of the tangent line to C at (x_0, y_0) which lies in the first quadrant. Then each point (x_0, y_0) of C is the midpoint of $L(x_0, y_0)$.



Problem 19 (Fa87, Sp92, Fa97, Sp00, Fa13, Sp19). Define a sequence of real numbers (x_n) by

$$x_0 = 1 \quad x_{n+1} = \frac{1}{2 + x_n} \quad \text{for } n \geq 0$$

Show that (x_n) converges and evaluate its limit.

Problem 20 (Fa87). Let S be the set of all real $C^1_{[0,1]}$ functions f on $[0, 1]$ such that $f(0) = 0$ and

$$\int_0^1 (f'(x))^2 dx \leq 1$$

Define

$$J(f) = \int_0^1 f(x) \, dx$$

Show that the function J is bounded on S , and compute its supremum. Is there a function $f_0 \in S$ at which J attains its maximum value? If so, what is f_0 ?