

Spring 2003

Problem 1 (Sp03). Let $\mathbb{F}$ be a field and $n \geq 1$. Prove that the following properties of an $n \times n$ matrix A with entries in $\mathbb{F}$ are equivalent:

- A is a scalar multiple of the identity matrix.
- Every nonzero vector $v \in \mathbb{F}^n$ is an eigenvector of A .

Problem 2 (Sp03). Define $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ by $f(x, 0) = 0$ and

$$f(x, y) = \left(1 - \cos \frac{x^2}{y}\right) \sqrt{x^2 + y^2}$$

for $y \neq 0$.

1. Show that f is continuous at $(0, 0)$.
2. Calculate all the directional derivatives of f at $(0, 0)$.
3. Show that f is not differentiable at $(0, 0)$.

Problem 3 (Sp03). Let $M_{2 \times 2}(\mathbb{Q})$ denote the ring of 2×2 matrices with entries in $\mathbb{Q}$. Let R be the set of matrices in $M_{2 \times 2}(\mathbb{Q})$ that commute with $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

1. Prove that R is a subring of $M_{2 \times 2}(\mathbb{Q})$.
2. Prove that R is isomorphic to the ring $\mathbb{Q}[x]/\langle x^2 \rangle$.

Problem 4 (Sp03, Fa23). Prove that for each integer $n \geq 0$ there is a polynomial $T_n(x)$ with integer coefficients such that the identity

$$2 \cos nz = T_n(2 \cos z)$$

holds for all z .

Problem 5 (Sp03). Let L be a real symmetric $n \times n$ matrix with 0 as a simple eigenvalue and $v \in \mathbb{R}^n$.

1. Show that for sufficiently small positive real ε , the equation $Lx + \varepsilon x = v$ has a unique solution $x = x(\varepsilon) \in \mathbb{R}^n$.
2. Evaluate $\lim_{\varepsilon \rightarrow 0^+} \varepsilon x(\varepsilon)$ in terms of v , the eigenvectors of L , and the inner product on $\mathbb{R}^n$.

Problem 6 (Sp03). Let x_n be a sequence of real numbers so that

$$\lim_{n \rightarrow \infty} (2x_{n+1} - x_n) = \alpha$$

Show that $\lim_{n \rightarrow \infty} x_n = \alpha$.

Problem 7 (Sp03). Let H_1 and H_2 be subgroups of a group G such that $H_1 \cup H_2$ is a subgroup of G .

1. Prove that either $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$.
2. Show that for each integer $n \geq 3$, there exists a group G with subgroups $H_1, H_2, \dots, H_n$, such that no H_i is contained in any other, and such that $H_1 \cup H_2 \cup \dots \cup H_n$ is a subgroup of G .

Problem 8 (Sp03). Evaluate $\int_0^\infty e^{-x^2} \cos x^2 dx$.

Problem 9 (Sp03). Let R be the set of complex numbers of the form $a + 3bi$, $a, b \in \mathbb{Z}$. Prove that R is a subring of $\mathbb{C}$, and that R is an integral domain but not a unique factorization domain.

Problem 10 (Sp03). 1. Prove that there is no continuously differentiable, measure preserving bijective function $f: \mathbb{R} \rightarrow \mathbb{R}_+$.

2. Find an example of a continuously differentiable, measure preserving bijective function $f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}_+$.

Problem 11 (Sp03). For an analytic function h on $\mathbb{C}$, let $h^{(i)}$ denote its i -th derivative, with $h^{(0)} = h$. Suppose that f and g are analytic functions on $\mathbb{C}$ satisfying

$$\begin{aligned} f^{(n)} + a_{n-1}f^{(n-1)} + \dots + a_0f^{(0)} &= 0 \\ g^{(m)} + b_{m-1}g^{(m-1)} + \dots + b_0g &= 0 \end{aligned}$$

for some constants $a_0, \dots, a_{n-1}, b_0, \dots, b_{m-1} \in \mathbb{C}$. Show that the product function $F = fg$ satisfies

$$c_{mn}F^{(mn)} + c_{mn-1}F^{(mn-1)} + \dots + c_0F = 0$$

for some constants $c_0, \dots, c_{mn} \in \mathbb{C}$ not all zero.

Problem 12 (Sp03, Fa23). Let f be an entire function such that $\Re f(z) \geq -2$ for all $z \in \mathbb{C}$. Show that f is constant.

Problem 13 (Sp03). Suppose G is a non-abelian simple group, and A is its automorphism group. Show that A contains a normal subgroup isomorphic to G .

Problem 14 (Su78, Sp03). Let X and Y be nonempty subsets of $\mathbb{R}^n$. Define

$$d(X, Y) = \inf\{d(x, y) \mid x \in X, y \in Y\}$$

1. Suppose X contains only one point x , and Y is closed. Prove $d(X, Y) = d(x, y)$ for some $y \in Y$.
2. Suppose X is compact and Y is closed. Prove $d(X, Y) = d(x, y)$ for some $x \in X, y \in Y$.
3. Show by example that the conclusion of Part 2 can be false if X and Y are closed but not compact.

Problem 15 (Sp03). Let $\text{GL}_2(\mathbb{C})$ denote the group of invertible 2×2 matrices with coefficients in the field of complex numbers. Let $\text{PGL}_2(\mathbb{C})$ denote the quotient of $\text{GL}_2(\mathbb{C})$ by the normal subgroup $\left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \mid \lambda \in \mathbb{C}^* \right\}$. Let n be a positive integer, and suppose that a, b are elements of $\text{PGL}_2(\mathbb{C})$ of order exactly n . Prove that there exists $c \in \text{PGL}_2(\mathbb{C})$ such that cac^{-1} is a power of b .

Problem 16 (Sp03). Let the function f be analytic in the unit disc $\mathbb{D}$. Suppose that $|f(z)| \leq 1$ in $\mathbb{D}$. Prove that if $f(z)$ has at least two fixed points z_1 and z_2 , then $f(z) = z$ for all $z \in \mathbb{D}$.

Problem 17 (Sp03). Let $N = 30030$, which is the product of the first six primes. How many nonnegative integers x less than N have the property that N divides $x^3 - 1$?

Problem 18 (Sp03). Let $A \subseteq \mathbb{R}$ be uncountable.

1. Show that A has at least one accumulation point.
2. Show that A has uncountably many accumulation points.