Spring 2004

Problem 1 (Sp04). Consider a sequence of functions $f_n : [a, b] \to \mathbb{R}$ with the property that for each $x \in [a, b]$ there is an open interval I_x containing x such that $\{f_n\}$ converges uniformly on $I_x \cap [a, b]$. Show that $\{f_n\}$ converges uniformly on $[a, b]$.

Problem 2 (Sp04). Find a countable abelian group whose endomorphism ring has the same cardinality as the set of real numbers.

Problem 3 (Sp04). Let $a_1, \ldots, a_n, b_1, \ldots, b_m$ be distinct complex numbers, $r_1, \ldots, r_n$ nonnegative integers, and $c_1, \ldots, c_m$ complex numbers. Prove that if $m \leqslant r_1 + \cdots + r_n + 1$, then there exists a rational function $F(z) \in \mathbb{C}(z)$ satisfying all of the following:

- $F(z)$ is holomorphic at ∞ and everywhere in $\mathbb{C}$ except possibly at $a_1, \ldots, a_n$.
- $\operatorname{ord}(F, a_i) \geqslant -r_i$.
- $F(b_j) = c_j$ for $j = 1, \ldots, m$.

Problem 4 (Sp04). For which positive integers n is it true that every invertible 2×2 matrix A with real entries can be expressed as the n-th power of another 2×2 matrix with real entries?

Problem 5 (Sp04). Suppose $f : \mathbb{R} \to \mathbb{C}$ satisfies $f'(t) + 2itf(t) = e^{2t}$ and $f(0) = 0$. Compute

$$\lim_{t \to +\infty} e^{it^2}(f(t) - f(-t))$$

Problem 6 (Sp04). For which pairs of integers (a, b) is the quotient ring

$$\mathbb{Z}[x]/\langle x^2 + ax + b \rangle$$

isomorphic to the direct product of rings $\mathbb{Z} \times \mathbb{Z}$?

Problem 7 (Sp04). Evaluate $\displaystyle\int_0^\infty \frac{\sin x}{x}\, \mathrm{d}x$.

Problem 8 (Sp04). Let V and W be finite dimensional vector spaces over a field $\mathbb{F}$. Let $f : V^n \to W$ be a function such that

- For each fixed $i \in \{1, \ldots, n\}$ and fixed $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n \in V$, the map from V to W

$$x \mapsto f(v_1, \ldots, v_{i-1}, x, v_{i+1}, \ldots, v_n)$$

  is a $\mathbb{F}$-linear transformation;

- $f(v_1, \ldots, v_n) = 0$ whenever $v_i = v_{i+1}$ for some $i \in \{1, \ldots, n-1\}$.

Prove that either $\dim V \geqslant n$ or f is identically zero.

Problem 9 (Sp04). Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable function. Prove that the following are equivalent:

(i) For every $x, y \in \mathbb{R}^n$, $(f(x) - f(y)) \cdot (x - y) \leqslant L\|x - y\|^2$.

(ii) For every $x, v \in \mathbb{R}^n$, $\mathrm{D}\, f(x)v \cdot v \leqslant L\|v\|^2$.

Problem 10 (Sp04). Let $\mathbb{F}$ be a field (of arbitrary characteristic). Suppose g is a nonnegative integer, and polynomials $a(x), b(x) \in \mathbb{F}[x]$ satisfy $\deg a(x) \leqslant g$ and $\deg b(x) = 2g + 1$. Prove that the polynomial $y^2 + a(x)y + b(x)$ is irreducible over $\mathbb{F}(x)$.

Problem 11 (Sp04). Find the maximum possible value of $|f'(1)|$ given that f is holomorphic on an open neighborhood of $\{z \in \mathbb{C} \mid |z| \leqslant 2\}$ and satisfies $|f(z)| \leqslant 1$ when $|z| = 2$.

Problem 12 (Sp04). Let A be a $d \times d$ matrix with complex entries. Assume that every eigenvalue of A has absolute value 1. Prove that there exists a constant $c \in \mathbb{R}$ independent of n such that

$$\|A^n x\| \leqslant c n^{d-1} \|x\|$$

for all $n \geqslant 1$ and $x \in \mathbb{C}^d$. Here $\|x\| = \sqrt{|x_1|^2 + \cdots + |x_d|^2}$ for all $(x_1, \ldots, x_d) \in \mathbb{C}^d$.

Problem 13 (Sp04). Let $a_1, \ldots, a_n$ be positive real numbers. Let Δ be the set of points $x \in \mathbb{R}^n$ satisfying the conditions

$$\sum_{i=1}^{n} a_i x_i = 1 \qquad x_i > 0 \text{ for all } i$$

Prove that the function $\log\left(\prod_{i=1}^{n} x_i\right)$ has a unique maximum on Δ and find the point where it occurs.

Problem 14 (Sp04). Let $n_1, \ldots, n_r$ be integers $\geqslant 2$. Prove that there is a finite group G containing elements $g_1, \ldots, g_r$ such that g_i has exact order n_i for each i, and $g_i g_j \neq g_j g_i$ for $i \neq j$.

Problem 15 (Sp04). Let $\{u_n\}$ be a sequence of functions that are defined and harmonic in an open neighborhood of the closed upper half-plane. Suppose that $\partial u_n / \partial y(x, 0) = 0$ for all $x \in \mathbb{R}$, and $u_n(x, 0)$ converges to 0 as $n \to \infty$ uniformly for $x \in \mathbb{R}$. Must $u_n(x, y) \to 0$ as $n \to \infty$ for every $(x, y) \in \mathbb{R} \times \mathbb{R}_+$?

Problem 16 (Sp04). Let A and B be $n \times n$ matrices with complex entries, such that $AB - BA$ is a linear combination of A and B. Prove that A and B have a common eigenvector.

Problem 17 (Sp04). For each real number x, compute $\lim_{n \to \infty} n \left(\left(1 + \frac{x}{n} \right)^n - e^x \right)$.

Problem 18 (Sp04). Determine the order of the automorphism group Aut_{S_4}.