Spring 2017

Problem 1 (Sp17). Show that the following improper Riemann integrals exist and are equal:

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x = \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\,\mathrm{d}x$$

Problem 2 (Sp17). Suppose f is a function from the reals to the reals satisfying $2f(x) = f(2x)$ for all x.

1. Prove that if f is differentiable at 0 then f is linear.
2. Give an example of such a function f that is continuous but not linear.

Problem 3 (Sp17). Suppose we have a continuous positive function $f : (0,\pi) \to (0,\infty)$ such that for all $x, y \in (0,\pi)$ we have

$$\int_x^y \frac{f(x)f(y)}{f^2(t)}\,\mathrm{d}t = \sin(y-x)$$

- Show that $\sin(z-x)f(y) = \sin(y-x)f(z) + \sin(z-y)f(x)$.
- Find all possibilities for f.

Problem 4 (Sp17). The Weierstrass zeta function ζ is a meromorphic function satisfying

- $\zeta(z+\omega_1) = \zeta(z) + \eta_1$
- $\zeta(z+\omega_2) = \zeta(z) + \eta_2$
- The singularities of ζ are poles of residue 1 at the points $m\omega_1 + n\omega_2$ for $m, n \in \mathbb{Z}$

Here $\omega_1, \omega_2, \eta_1, \eta_2$ are complex constants with ω_2/ω_1 not real. Prove Legendre's relation $\omega_2\eta_1 - \omega_1\eta_2 = \pm 2\pi i$ and express the sign in terms of ω_1 and ω_2.

Problem 5 (Fa94, Sp17). Suppose the coefficients of the power series $\sum_{n=0}^{\infty} a_n z^n$ are given by the recurrence relation

$$a_0 = 1 \quad a_1 = -1 \quad 3a_n + 4a_{n-1} - a_{n-2} = 0 \qquad n = 2, 3, \ldots$$

Find the radius of convergence of the series and the function to which it converges in its disc of convergence.

Problem 6 (Sp97, Sp17). Show that

$$\det e^M = e^{\operatorname{tr} M}$$

for any complex $n \times n$ matrix M, where e^M is defined as in **??**.

Problem 7 (Sp17). Given two vectors x and y in $\mathbb{R}^n$ with $\|x\| = \|y\|$, construct an orthogonal matrix Q such that $Qx = y$. Can there be such a matrix if $\|x\| \neq \|y\|$?

Problem 8 (Sp17, Sp18). Show that for each integer $p \geqslant 0$, $S_p(n) = \sum_{k=0}^{n} k^p$ is a polynomial of degree $p+1$ in the variable n.

Problem 9 (Sp17). The Bell number P_n is the number of partitions of a set of n elements into disjoint nonempty subsets, so for example $\{1,2,3\} = \{1\} \cup \{2\} \cup \{3\} = \{1,2\} \cup \{3\} = \{2,3\} \cup \{1\} = \{1,3\} \cup \{2\}$ and $P_3 = 5$. Show that

$$\frac{P_n}{n!} \to 0$$

as $n \to \infty$.

Problem 10 (Sp17). Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ with the property that

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h}$$

for all $x \in \mathbb{R}$ and all $h \neq 0$.

Problem 11 (Sp88, Sp10, Sp17). Does there exist a continuous real valued function $f(x)$, $0 \leqslant x \leqslant 1$, such that

$$\int_0^1 x f(x)\,\mathrm{d}x = 1 \quad \text{and} \quad \int_0^1 x^n f(x)\,\mathrm{d}x = 0$$

for $n = 0, 2, 3, 4, \ldots$? Give an example or a proof that no such f exists.

Problem 12 (Sp17). The error of a quadrature rule with $p+1$ distinct points x_j, weights w_j is

$$E_p(f) = \int_a^b f(x)\,\mathrm{d}x - \sum_{j=0}^{p} w_j f\left(x_j\right)$$

Suppose that $E_p(f) = 0$ whenever f is a polynomial of degree $\leqslant q$. Show that $q \leqslant 2p+1$ and if $q \geqslant 2p$ then $w_j > 0$ for all j.

Problem 13 (Sp17). Given n distinct points $z_j \in \mathbb{C}$ and n values $f_j \in \mathbb{C}$, show that there is a unique polynomial P of degree at most $n-1$ such that

$$P\left(z_j\right) = f_j$$

for $1 \leqslant j \leqslant n$.

Problem 14 (Su77, Sp17). Write all values of i^i in the form $a + bi$.

Problem 15 (Sp15, Fa16, Sp17). Let $\mathbb{D}$ be the unit disc in the complex plane $\mathbb{C}$, $f : \mathbb{D} \to \mathbb{C}$ an analytic function with

$$|f^{(k)}(0)| \leqslant M$$

for all $k \geqslant 0$, and let $t_i, s_i \in \mathbb{D}$ for $1 \leqslant i \leqslant n$. Define the matrix $A_{ij} = f\left(t_i s_j\right)$ for $1 \leqslant i, j \leqslant n$. For each $r \geqslant 1$ find an $n \times n$ matrix B with rank $\leqslant r$ and

$$|A_{ij} - B_{ij}| \leqslant \frac{2M}{r!}$$

for $1 \leqslant i, j \leqslant n$.

Problem 16 (Sp17). Suppose R is an invertible upper triangular complex matrix and A is symmetric. Find an explicit formula for the entries of the upper triangular matrix E satisfying

$$E^{\mathsf{T}} R + R^{\mathsf{T}} E = A$$

and show that your solution is unique.

Problem 17 (Sp12, Sp16, Sp17). Find the structure of the group of units of $\mathbb{Z}_{2012}$.

Problem 18 (Sp80, Sp17).

1. Exhibit an element of S_9 of order 20.
2. Prove that no element of S_9 has order 18.