Spring 2018

Problem 1 (Sp18). Calculate $\tan 0.1$ to 4 decimal places.

Problem 2 (Sp18, Fa19). Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that

$$\lim_{n\to\infty} \int_0^1 f(x) \sin nx \, \mathrm{d}x = 0$$

Problem 3 (Sp18). Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function that satisfies the differential equation

$$f''(x) = g(x) f'(x)$$

for some strictly positive smooth function g. Prove that f is either strictly increasing, strictly decreasing, or constant.

Problem 4 (Fa77, Su82, Fa97, Sp01, Fa03, Fa08, Sp11, Fa12, Sp18, Sp21). Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^{2n}}$, where n is a positive integer.

Problem 5 (Sp83, Sp18). Determine all the complex analytic functions f defined on the unit disc $\mathbb{D}$ which satisfy

$$f''\left(\frac{1}{n}\right) + f\left(\frac{1}{n}\right) = 0$$

for $n = 2, 3, 4, \ldots$

Problem 6 (Sp80, Sp10, Fa13, Sp18). Let A and B be $n \times n$ complex matrices. Prove or disprove each of the following statements:

1. If A and B are diagonalizable, so is $A + B$.
2. If A and B are diagonalizable, so is AB.
3. If $A^2 = A$, then A is diagonalizable.
4. If A is invertible and A^2 is diagonalizable, then A is diagonalizable.
5. If AB is diagonalizable and invertible then so is BA.

Problem 7 (Sp18). Let $A, B \in \mathbb{R}^{n\times n}$ be two symmetric matrices with positive eigenvalues. Show that there is an invertible matrix $C \in \mathbb{R}^{n\times n}$ such that

$$C^{\mathsf{T}} A C = \left(C^{\mathsf{T}} B C\right)^{-1}$$

Problem 8 (Sp18). Let R be the ring of complex numbers of the form $m + n\sqrt{-2}$ with m, n integers. Prove that if $a, b \in R$ with $b \neq 0$ then we can find $q, r \in R$ such that $a = bq + r$ and $|r| < |b|$. Give an example to show this is false for the ring of numbers $m + n\sqrt{-3}$.

Problem 9 (Sp18). Find the largest possible order of an element of S_{12}.

Problem 10 (Sp18). Let N be a positive integer and $\omega = e^{2\pi i/N}$.

1. Show that the $N \times N$ matrix F with entries $F_{pq} = \dfrac{\omega^{pq}}{\sqrt{N}}$ is unitary.
2. For each $x \in \mathbb{R}$ and each integer k with $0 \leqslant k \leqslant N - 1$, find an explicit formula for the sum of the series

$$E_k(x) = \sum_{p=0}^{\infty} \frac{x^{pN+k}}{(pN + k)!}$$

Problem 11 (Sp09, Sp18). Suppose that f is a continuous real function on $[0, 1]$. Prove that

$$\lim_{\alpha \to 0^+} \alpha \int_0^1 x^{\alpha - 1} f(x)\, \mathrm{d}x = f(0)$$

Problem 12 (Sp18). Prove that the sequence of functions $f_n(x) = \sin nx$ has no uniformly convergent subsequence on compact subsets of $\mathbb{R}$.

Problem 13 (Sp18). Find $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n}$.

Problem 14 (Sp78, Sp18, Fa21). Show that there is a complex analytic function defined on the set $U = \{z \in \mathbb{C} \mid |z| > 4\}$ whose derivative is

$$f(z) = \frac{z}{(z-1)(z-2)(z-3)}$$

Is there a complex analytic function on U whose derivative is

$$g(z) = \frac{z^3}{(z-1)(z-2)(z-3)}\ ?$$

Problem 15 (Sp17, Sp18). Show that for each integer $p \geqslant 0$, $S_p(n) = \displaystyle\sum_{k=0}^{n} k^p$ is a polynomial of degree $p + 1$ in the variable n.

Problem 16 (Sp90, Fa93, Sp13, Sp18, Sp19, Sp22). 1. If A is a matrix over an algebraically closed field of characteristic 0 with $A^n = I$ for some $n > 0$ show that A is diagonalizable.

2. Show that over any field of characteristic $p > 0$ there is a matrix A, with $A^n = I$ for some $n > 0$, that is not diagonalizable.

Problem 17 (Sp18). Let G be an abelian group. For any integer m, denote by $G[m]$ the subgroup of elements of G whose orders divide m. Let p be a prime number. Assume that $G[p]$ has order p^r for some integer $r \geqslant 0$. Prove that the order of $G[p^n]$ is at most p^{rn} for any integer $n \geqslant 1$.

Problem 18 (Sp18). Find the smallest possible order of a non-abelian group of odd order.

Problem 19 (Sp18). Suppose A and B are invertible $n \times n$ matrices of complex numbers. Given $AB = S\Lambda S^{-1}$, find T such that $BA = T\Lambda T^{-1}$.