

Spring 2019

Problem 1 (Sp95, Sp19). Let y be a solution of $y''' - y = 0$ such that $y(t) \rightarrow 0$ as $t \rightarrow \infty$. Show that $y''(0) + y'(0) + y(0) = 0$.

Problem 2 (Sp95, Sp19). Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a bounded continuously differentiable function. Show that every solution of $y'(x) = f(y(x))$ is monotone.

Problem 3 (Sp19). Let f be a twice continuously differentiable function on $[0, 1]$ such that $f(0) = f(1) = 0$. Prove that

$$\max_{x \in [0,1]} |f(x)| \leq \frac{1}{8} \max_{x \in [0,1]} |f''(x)|$$

and find an example where equality holds.

Problem 4 (Sp81, Sp10, Sp19, Sp22). Evaluate $\int_{-\infty}^{\infty} \frac{x \sin x}{(1 + x^2)^2} dx$.

Problem 5 (Sp10, Sp19). How many complex numbers z are there such that $|z| < 2$, $\Im(z) > 0$, and $z^7 + e^z = 0$?

Problem 6 (Sp19). Let n be a positive integer and let a be a complex number. Prove that $a^n = 1$ if and only if there are invertible $n \times n$ complex matrices X, Y such that $XY = aYX$.

Problem 7 (Sp90, Fa93, Sp13, Sp18, Sp19, Sp22). 1. If A is a matrix over an algebraically closed field of characteristic 0 with $A^n = I$ for some $n > 0$ show that A is diagonalizable.

2. Show that over any field of characteristic $p > 0$ there is a matrix A , with $A^n = I$ for some $n > 0$, that is not diagonalizable.

Problem 8 (Fa77, Fa96, Sp19). Let $\mathbb{C}^3$ denote the set of ordered triples of complex numbers. Define a map $F : \mathbb{C}^3 \rightarrow \mathbb{C}^3$ by

$$F(u, v, w) = (u + v + w, uv + vw + wu, uvw)$$

Prove that F is onto but not one-to-one.

Problem 9 (Sp19). Let S be a countable set of real numbers. Show that there are functions $g_n : \mathbb{N} \rightarrow \mathbb{N}$ such that if $f : \mathbb{N} \rightarrow \mathbb{N}$ is a function with $f(n+1) > g_n(f(n))$ for all n , then $A = \sum_{n=1}^{\infty} 1/f(n)$ converges to a real that is not in the set S .

Problem 10 (Sp19). Evaluate $\int_0^{\pi/2} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx$.

Problem 11 (Sp19). For $t \geq 0$ let

$$F(t) = \int_0^t \exp(-x^2) dx \quad \text{and} \quad G(t) = \int_0^1 \frac{\exp(-t^2(1+x^2))}{1+x^2} dx$$

Show that $F(t)^2 + G(t)$ is constant and deduce the value of $F(\infty)$.

Problem 12 (Fa87, Sp92, Fa97, Sp00, Fa13, Sp19). Define a sequence of real numbers (x_n) by

$$x_0 = 1 \quad x_{n+1} = \frac{1}{2 + x_n} \quad \text{for } n \geq 0$$

Show that (x_n) converges and evaluate its limit.

Problem 13 (Sp19). Let $c_0, c_1, \dots, c_{n-1}$ be complex numbers. Prove that all the zeros of the polynomial

$$z^n + c_{n-1}z^{n-1} + \dots + c_1z + c_0$$

lie in the open disc with center 0 and radius

$$1 + |c_{n-1}| + \dots + |c_1| + |c_0|$$

Problem 14 (Fa87, Sp19). If $f(z)$ is analytic in the open disc $|z| < 1$, and $|f(z)| \leq \frac{1}{1-|z|}$. Show that

$$\left| \frac{f^{(n)}(0)}{n!} \right| \leq (n+1) \left(1 + \frac{1}{n} \right)^n < e(n+1)$$

Problem 15 (Sp19). Let $\mathbb{Z}_2$ be the ring of integers mod 2. Prove the following identity in $\mathbb{Z}_2[x_1, \dots, x_n]$:

$$\det \begin{pmatrix} x_1 & & x_n \\ x_1^2 & & x_n^2 \\ & \ddots & \\ x_1^{2^{n-1}} & & x_n^{2^{n-1}} \end{pmatrix} = \prod (a_1x_1 + \dots + a_nx_n)$$

where $(a_1, \dots, a_n)$ run all nonzero vectors in $\mathbb{Z}_2^n$.

Problem 16 (Sp19). Is it true that elements of the group $\text{GL}_2^+(\mathbb{R})$ of real 2×2 matrices with positive determinant are conjugate in $\text{GL}_2^+(\mathbb{R})$ if and only if the matrices are similar, i.e., conjugate in $\text{GL}_2(\mathbb{R})$?

Problem 17 (Sp19). Let R be a ring in which every element is idempotent. Show that R has characteristic 2 and is commutative.

Problem 18 (Sp19). Let A_6 be the alternating group on 6 letters. Prove or give a counterexample to the statement: For every $\sigma \in A_6$ there is a $\tau \in S_6$ such that $\tau^2 = \sigma$.