Spring 2023

Problem 1 (Sp23). Evaluate the infinite product $\displaystyle\prod_{n=3}^{\infty} \frac{n^2-4}{n^2-1}$.

Problem 2 (Sp89, Fa06, Sp23). Let $f : \mathbb{R} \times [0,1] \to \mathbb{R}$ be a continuous function. For $x \in \mathbb{R}$, define

$$g(x) = \max\{f(x,y) \mid y \in [0,1]\}$$

Show that g is continuous.

Problem 3 (Sp23). Prove Taylor's theorem with the remainder on the form of Peano: If a real valued function on the number line has a well-defined n-th derivative at $x = 0$, then the error of approximating the function near $x = 0$ by its degree n Taylor polynomial is $o(x^n)$ when $x \to 0$.

Problem 4 (Sp23). Suppose that f is a complex polynomial all of whose zeros have real part at most 0. Show that if $r > 0$ then $|f(r)| \geqslant |f(-r)|$. Give an example to show that the conclusion can be false if the condition that f is a polynomial is replaced by the condition that f is entire.

Problem 5 (Sp23). Evaluate the contour integral $\displaystyle\lim_{R\to+\infty} \int_{c-iR}^{c+iR} \frac{1}{z}\,\mathrm{d}z$ where c is a nonzero real number.

Problem 6 (Sp23). A plane passing through the origin in $\mathbb{R}^3$ intersects the ellipsoid $x^2/4 + y^2/9 + z^2/16 = 1$ by an ellipse. Determine how many such sections are circles and find their radii.

Problem 7 (Sp23). Suppose that the square complex matrix A is similar to A^n for some integer $n > 1$. Prove all eigenvalues of A are either zero or roots of unity.

Problem 8 (Sp23). Let $\alpha : G \to G_1$ and $\beta : G \to G_2$ be group homomorphisms.

1. Show that if $\ker\alpha \subseteq \ker\beta$ and α is surjective then there is a well-defined group homomorphism $\varphi : G_1 \to G_2$ such that $\beta = \varphi \circ \alpha$.
2. Show that if $\ker\alpha \not\subseteq \ker\beta$ then there is no such homomorphism φ.

Problem 9 (Sp23). Prove or give a counterexample: For every $\sigma \in A_5$ there is a $\tau \in S_5$ such that $\tau^2 = \sigma$.

Problem 10 (Sp23). Let $f\colon (a, b] \to \mathbb{R}$ be a function. Assume that f is strictly increasing on (a, b). Assume also that f is continuous from the left at b. Then show that f is strictly increasing on $(a, b]$.

Problem 11 (Sp23). The *unit cube* in the space $\mathcal{C}_{[0,1]}$ of continuous real valued functions on the closed interval $[0, 1]$ consists of those functions whose norm $\|f\|_\infty = \max_{0 \leqslant t \leqslant 1} |f(t)|$ doesn't exceed 1. Find a linear map $\mathbb{R}^3 \to \mathcal{C}_{[0,1]}$ such that the inverse image of the unit cube is the unit ball in $\mathbb{R}^3$.

Problem 12 (Sp23). Show that the recursive sequence $x_{n+1} = x_n/2 + 1/x_n$ with the initial value $x_0 = 1.5$ converges to $\sqrt{2}$, and that x_{10} is within 10^{-1000} of $\sqrt{2}$.

Problem 13 (Sp23). Describe a holomorphic function on the open unit disc that is a bijection to the vertical strip $0 < \Re z < 1$.

Problem 14 (Sp23). Find the radius of convergence of the Taylor series of $\dfrac{1}{e^x + e^{-x}}$ at the point $x = 1$.

Problem 15 (Fa81, Sp84, Sp23). If two real matrices are similar over $\mathbb{C}$, then they are similar over $\mathbb{R}$.

Problem 16 (Sp23). Give an example of two square complex matrices that have the same minimal polynomial and the same characteristic polynomial but are not similar.

Problem 17 (Sp23). Let n be a positive integer, and p any prime. Prove that over a finite field $\mathbb{F}$ of $p^{\varphi(n)}$ elements the polynomial $x^n - 1$ factors into linear factors.

Problem 18 (Sp23). Give an example of a commutative ring which has an infinite descending chain of distinct prime ideals $\mathfrak{I}_1 \supset \mathfrak{I}_2 \cdots \supset \mathfrak{I}_n \supset \cdots$