Spring 1979

Problem 1 (Sp79). Let $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be differentiable. Suppose

$$\lim_{x \to 0} \frac{\partial f}{\partial x_j}(x)$$

exists for each $j = 1, \ldots, n$.

1. Can f be extended to a continuous map from $\mathbb{R}^n$ to $\mathbb{R}$?
2. Assuming continuity at the origin, is f differentiable from $\mathbb{R}^n$ to $\mathbb{R}$?

Problem 2 (Sp79). Let E denote a finite dimensional complex vector space with a Hermitian inner product $\langle x, y \rangle$.

1. Prove that E has an orthonormal basis.
2. Let $f : E \times E \to \mathbb{C}$ be such that $f(x, y)$ is linear in x and conjugate linear in y. Show there is a linear map $A_f : E \to E$ such that $f(x, y) = \langle A_f x, y \rangle$.

Problem 3 (Sp79). Find all n such that some element of S_7 has order n.

Problem 4 (Sp79). Prove that every compact metric space has a countable dense subset.

Problem 5 (Sp79). Find all solutions to the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{y} \qquad y(0) = 0$$

Problem 6 (Sp79, Su85, Sp89, Fa19). Prove that if $1 < \lambda < \infty$, the function

$$f_\lambda(z) = z + \lambda - e^z$$

has only one zero in the half-plane $\Re z < 0$, and that this zero is real.

Problem 7 (Sp79). Evaluate $\displaystyle\int_0^\infty \frac{x^2 + 1}{x^4 + 1}\,\mathrm{d}x$.

Problem 8 (Sp79). Let M be a real nonsingular 3×3 matrix. Prove there are real matrices S and U such that $M = SU = US$, all the eigenvalues of U equal 1, and S is diagonalizable over $\mathbb{C}$.

Problem 9 (Sp79). Let M be an $n \times n$ complex matrix. Let G_M be the set of complex numbers λ such that the matrix λM is similar to M.

1. What is G_M if
$$M = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}?$$
2. Assume M is *not* nilpotent. Prove G_M is finite.

Problem 10 (Sp79). Let p be a prime and $f(x) \in \mathbb{Z}_p[x]$. Let $g(x) = x^p - x$. Show that the greatest common divisor of $f(x)$ and $g(x)$ is the product of the distinct linear factors of $f(x)$.

Problem 11 (Sp79, Sp97, Sp24). Classify all abelian groups of order 80 up to isomorphism.

Problem 12 (Sp79, Fa81). Let G be a group with three normal subgroups N_1, N_2, and N_3. Suppose $N_i \cap N_j = \{e\}$ and $N_i N_j = G$ for all i, j with $i \neq j$. Show that G is abelian and N_i is isomorphic to N_j for all i, j.

Problem 13 (Sp79). Consider the system of differential equations:
$$\frac{\mathrm{d}x}{\mathrm{d}t} = y + tz \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = z + t^2 x \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = x + e^t y$$
Prove there exists a solution defined for all $t \in [0, 1]$, such that
$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
and also
$$\int_0^1 \left(x(t)^2 + y(t)^2 + z(t)^2\right) \mathrm{d}t = 1$$

Problem 14 (Sp79, Fa93). Let $M_{n\times n}$ denote the vector space of $n \times n$ real matrices for $n \geqslant 2$. Let $\det : M_{n\times n} \to \mathbb{R}$ be the determinant map.

1. Show that det is $\mathcal{C}^\infty$.
2. Show that the derivative of det at $A \in M_{n\times n}$ is zero if and only if A has rank $\leqslant n - 2$.

Problem 15 (Fa78, Sp79). Which of the following matrices are similar as matrices over $\mathbb{R}$?

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad A_5 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad A_6 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Problem 16 (Sp79). For which $z \in \mathbb{C}$ does $\displaystyle\sum_{n=0}^{\infty} \left(\frac{z^n}{n!} + \frac{n^2}{z^n} \right)$ converge?

Problem 17 (Sp79, Sp83). Let P and Q be complex polynomials with the degree of Q at least two more than the degree of P. Prove there is an $r > 0$ such that if C is a closed curve outside $|z| = r$, then

$$\int_C \frac{P(z)}{Q(z)} \, \mathrm{d}z = 0$$

Problem 18 (Sp79). Show that for any continuous function $f : [0, 1] \to \mathbb{R}$ and $\varepsilon > 0$, there is a function of the form

$$g(x) = \sum_{k=0}^{n} C_k x^{4k}$$

for some $n \in \mathbb{Z}$, where $C_0, \ldots, C_n \in \mathbb{Q}$ and $|g(x) - f(x)| < \varepsilon$ for all x in $[0, 1]$.

Problem 19 (Sp79). Let P be a 9×9 real matrix such that $x^{\mathsf{T}} P y = -y^{\mathsf{T}} P x$ for all column vectors x, y in $\mathbb{R}^9$. Prove that P is singular.

Problem 20 (Sp79). Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ having all three of the following properties:

- $f(x) = 0$ for $x < 0$ and $x > 2$,
- $f'(1) = 1$,
- f has derivatives of all orders.