Spring 1985

Problem 1 (Sp85, Fa10). Let $f(x)$, $0 \leqslant x < \infty$, be continuous, differentiable, with $f(0) = 0$, and that $f'(x)$ is an increasing function of x for $x \geqslant 0$. Prove that

$$g(x) = \begin{cases} f(x)/x & x > 0 \\ f'(0) & x = 0 \end{cases}$$

is an increasing function of x.

Problem 2 (Sp85). In a abelian group G, let the element a have order r, let b have order s $(r, s < \infty)$, and assume that the greatest common divisor of r and s is 1. Show that ab has order rs.

Problem 3 (Sp85). Show that a necessary and sufficient condition for three points a, b, and c in the complex plane to form an equilateral triangle is that

$$a^2 + b^2 + c^2 = bc + ca + ab$$

Problem 4 (Sp85). Let $R > 1$ and let f be analytic on $|z| < R$ except at $z = 1$, where f has a simple pole. If

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad \text{for} \quad |z| < 1$$

is the Maclaurin series for f, show that $\lim\limits_{n\to\infty} a_n$ exists.

Problem 5 (Sp85). Factor $x^4 + x^3 + x + 3$ completely in $\mathbb{Z}_5[x]$.

Problem 6 (Sp85, Fa88). Let A and B be two $n \times n$ self-adjoint, i.e., Hermitian, matrices over $\mathbb{C}$ such that all eigenvalues of A lie in $[a, a']$ and all eigenvalues of B lie in $[b, b']$. Show that all eigenvalues of $A + B$ lie in $[a + b, a' + b']$.

Problem 7 (Sp85). Prove that $\displaystyle\int_0^{\infty} e^{-x^2} \cos 2bx \, \mathrm{d}x = \frac{1}{2}\sqrt{\pi} e^{-b^2}$. What restrictions, if any, need be placed on b?

Problem 8 (Sp85). Let $h > 0$ be given. Consider the linear difference equation

$$\frac{y((n+2)h) - 2y((n+1)h) + y(nh)}{h^2} = -y(nh) \qquad n = 0, 1, 2, \ldots$$

1. Find the general solution of the equation by trying suitable exponential substitutions.

2. Find the solution with $y(0) = 0$ and $y(h) = h$. Denote it by $S_h(nh)$, $n = 1, 2, \ldots$.

3. Let x be fixed and $h = \dfrac{x}{n}$. Show that $\lim\limits_{n\to\infty} S_{x/n}\left(\dfrac{nx}{n}\right) = \sin x$

Problem 9 (Sp85). Define the function ζ by $\zeta(x) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^x}$. Prove that $\zeta(x)$ is defined and has continuous derivatives of all orders in the interval $1 < x < \infty$.

Problem 10 (Sp85). For arbitrary elements a, b, and c in a field $\mathbb{F}$, compute the minimal polynomial of the matrix

$$\begin{pmatrix} 0 & 0 & a \\ 1 & 0 & b \\ 0 & 1 & c \end{pmatrix}$$

Problem 11 (Sp85). Let $\mathbb{F} = \left\{a + b\sqrt[3]{2} + c\sqrt[3]{4} \mid a, b, c \in \mathbb{Q}\right\}$. Prove that $\mathbb{F}$ is a field and each element in $\mathbb{F}$ has a unique representation as $a + b\sqrt[3]{2} + c\sqrt[3]{4}$ with $a, b, c \in \mathbb{Q}$. Find $\left(1 - \sqrt[3]{2}\right)^{-1}$ in $\mathbb{F}$.

Problem 12 (Fa79, Fa80, Sp85, Su85, Fa96, Fa98, Sp99, Sp26). Prove that

$$\int_0^{\infty} \frac{x^{\alpha-1}}{1+x}\,\mathrm{d}x = \frac{\pi}{\sin \pi\alpha}$$

What restrictions must be placed on α?

Problem 13 (Sp85). Prove that for any $a \in \mathbb{C}$ and any integer $n \geqslant 2$, the equation $1 + z + az^n = 0$ has at least one root in the disc $|z| \leqslant 2$.

Problem 14 (Sp85). Show that $I = \displaystyle\int_0^{\pi} \log \sin x\,\mathrm{d}x$ converges as an improper Riemann integral. Evaluate I.

Problem 15 (Sp85, Sp11, Fa17, Sp21). Find an irreducible polynomial over the integers with $2\cos(2\pi/7)$ as a zero, and use this to show that it is not contained in any extension of the rational numbers of degree a power of 2.

Problem 16 (Sp85). Let f be continuous on $\mathbb{R}$, and let

$$f_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} f\left(x + \frac{k}{n}\right)$$

Prove that $f_n(x)$ converges uniformly to a limit on every finite interval $[a, b]$.

Problem 17 (Sp85). Let v_1 and v_2 be two real valued continuous functions on $\mathbb{R}$ such that $v_1(x) < v_2(x)$ for all $x \in \mathbb{R}$. Let $\varphi_1(t)$ and $\varphi_2(t)$ be, respectively, solutions of the differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v_1(x) \quad \text{and} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = v_2(x)$$

for $a < t < b$. If $\varphi_1(t_0) = \varphi_2(t_0)$ for some $t_0 \in (a, b)$, show that $\varphi_1(t) \leqslant \varphi_2(t)$ for all $t \in (t_0, b)$.

Problem 18 (Sp85, Sp05). Let A and B be two $n \times n$ self-adjoint, i.e., Hermitian matrices over $\mathbb{C}$ and assume A is positive definite. Prove that all eigenvalues of AB are real.

Problem 19 (Sp85). Let $\mathbb{F}$ be a finite field. Give a complete proof of the fact that the number of elements of $\mathbb{F}$ is of the form p^r, where $p \geqslant 2$ is a prime number and r is an integer $\geqslant 1$.

Problem 20 (Sp85). Let $f(z)$ be an analytic function that maps the open disc $|z| < 1$ into itself. Show that

$$|f'(z)| \leqslant \frac{1}{1 - |z|^2}$$