

Summer 1984

Problem 1 (Su84). Show that if a subgroup H of a group G has just one left coset different from itself, then it is a normal subgroup of G .

Problem 2 (Su84). Let $\mathbb{Z}$ be the ring of integers and $\mathbb{Z}[x]$ the polynomial ring over $\mathbb{Z}$. Show that

$$x^6 + 539x^5 - 511x + 847$$

is irreducible in $\mathbb{Z}[x]$.

Problem 3 (Su84). Let $f: \mathbb{R}^m \rightarrow \mathbb{R}^n$, $n \geq 2$, be a linear transformation of rank $n - 1$. Let $f(v) = (f_1(v), f_2(v), \dots, f_n(v))$ for $v \in \mathbb{R}^m$. Show that a necessary and sufficient condition for the system of inequalities $f_i(v) > 0$, $i = 1, \dots, n$, to have no solution is that there exist real numbers $\lambda_i \geq 0$, not all zero, such that

$$\sum_{i=1}^n \lambda_i f_i = 0$$

Problem 4 (Su84, Fa23). Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a real matrix with $a, b, c, d > 0$. Show that A has an eigenvector $(x, y) \in \mathbb{R}^2$ with $x, y > 0$.

Problem 5 (Su84). 1. Show that there is a unique analytic branch outside the unit circle of the function $f(z) = \sqrt{z^2 + z + 1}$ such that $f(t)$ is positive when $t > 1$.

2. Using the branch determined in Part 1, calculate the integral

$$\frac{1}{2\pi i} \int_{C_r} \frac{dz}{\sqrt{z^2 + z + 1}}$$

where C_r is the positively oriented circle $|z| = r$ and $r > 1$.

Problem 6 (Su84, Fa23). Let $\rho > 0$. Show that for n large enough, all the zeros of

$$f_n(z) = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots + \frac{1}{n!z^n}$$

lie in the circle $|z| < \rho$.

Problem 7 (Su84). Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be $\mathcal{C}^1$ and let

$$\begin{aligned}u &= f(x) \\ v &= -y + xf(x)\end{aligned}$$

If $f'(x_0) \neq 0$, show that this transformation is locally invertible near (x_0, y_0) and the inverse has the form

$$\begin{aligned}x &= g(u) \\ y &= -v + ug(u)\end{aligned}$$

Problem 8 (Su84, Fa23). Let $\varphi(s)$ be a $\mathcal{C}^2$ function on $[1, 2]$ with φ and φ' vanishing at $s = 1, 2$. Prove that there is a constant $C > 0$ such that for any $\lambda > 1$,

$$\left| \int_1^2 e^{i\lambda x} \varphi(x) dx \right| \leq \frac{C}{\lambda^2}$$

Problem 9 (Su84, Fa23). Consider the solution curve $(x(t), y(t))$ to the equations

$$\begin{aligned}\frac{dx}{dt} &= 1 + \frac{1}{2}x^2 \sin y \\ \frac{dy}{dt} &= 3 - x^2\end{aligned}$$

with initial conditions $x(0) = 0$ and $y(0) = 0$. Prove that the solution must cross the line $x = 1$ in the xy plane by the time $t = 2$.

Problem 10 (Su84). Let $\mathcal{C}^{1/3}$ be the set of real valued functions f on the closed interval $[0, 1]$ such that

- $f(0) = 0$
- $\|f\|$ is finite, where by definition

$$\|f\| = \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|^{1/3}} \mid x \neq y \right\}$$

Verify that $\|\cdot\|$ is a norm for the space $\mathcal{C}^{1/3}$, and prove that $\mathcal{C}^{1/3}$ is complete with respect to this norm.

Problem 11 (Su84). Find four different subgroups of S_4 isomorphic to S_3 and nine isomorphic to S_2 .

Problem 12 (Su79, Sp82, Sp83, Su84, Fa91, Fa98, Sp11, Fa15, Fa16, Fa23). Let $\mathbb{F}$ be a finite field with q elements and let V be an n -dimensional vector space over $\mathbb{F}$.

1. Determine the number of elements in V .
2. Let $\text{GL}_n(\mathbb{F})$ denote the group of all $n \times n$ nonsingular matrices over $\mathbb{F}$. Determine the order of $\text{GL}_n(\mathbb{F})$.
3. Let $\text{SL}_n(\mathbb{F})$ denote the subgroup of $\text{GL}_n(\mathbb{F})$ consisting of matrices with determinant 1. Find the order of $\text{SL}_n(\mathbb{F})$.

Problem 13 (Su84). Let A be a 2×2 matrix over $\mathbb{C}$ which is not a scalar multiple of the identity matrix I . Show that any 2×2 matrix X over $\mathbb{C}$ commuting with A has the form $X = \alpha I + \beta A$, where $\alpha, \beta \in \mathbb{C}$.

Problem 14 (Su84). Suppose V is an n -dimensional vector space over the field $\mathbb{F}$. Let $W \subset V$ be a subspace of dimension $r < n$. Show that

$$W = \cap \{U \mid U \text{ is an } (n-1) \text{ - dimensional subspace of } V \text{ and } W \subset U\}$$

Problem 15 (Su84). Let $\mathbb{Z}_3$ be the field of integers mod 3 and $\mathbb{Z}_3[x]$ the corresponding polynomial ring. Decompose $x^3 + x + 2$ into irreducible factors in $\mathbb{Z}_3[x]$.

Problem 16 (Su84, Fa18). Let $p(z)$ be a nonconstant polynomial with real coefficients such that for some real number a , $p(a) \neq 0$ but $p'(a) = p''(a) = 0$. Prove that the equation $p(z) = 0$ has a nonreal root.

Problem 17 (Su84). Suppose

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

has radius of convergence $R > 0$. Show that

$$h(z) = \sum_{n=0}^{\infty} \frac{a_n z^n}{n!}$$

is entire and that for $0 < r < R$, there is a constant M such that $|h(z)| \leq M e^{|z|/r}$.

Problem 18 (Su84, Fa19). Show there is a unique continuous real valued function $f : [0, 1] \rightarrow \mathbb{R}$ such that

$$f(x) = \sin x + \int_0^1 \frac{f(y)}{e^{x+y}} dy$$

Problem 19 (Su84). Let $x(t)$ be the solution of the differential equation

$$x''(t) + 8x'(t) + 25x(t) = 2 \cos t$$

with initial conditions $x(0) = 0$ and $x'(0) = 0$. Show that for suitable constants α and δ ,

$$\lim_{t \rightarrow \infty} (x(t) - \alpha \cos(t - \delta)) = 0$$

Problem 20 (Su84, Fa13). Evaluate $\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 4x + 20} dx$.